Lecture 13

- Recap
- Covariant and gauge-invariant form of ME
- Covariant formulation of Newtonian mechanics
- Proper time, 4-velocity and 4-acceleration
- Lorentz force, 4-momentum and energy.
  ($E = mc^2$)

Applications:

- Constant electric field
- Constant magnetic field

Recap

$$v_p' = \frac{v_p - v}{1 - \frac{v v_p}{c^2}}$$

$$s^2 = -c^2 \Delta t^2 + (\Delta \vec{x})^2 = \Delta x^\mu \Delta x_\mu$$

$$\Box \begin{pmatrix} \varphi \\ c\vec{A} \end{pmatrix} = \frac{1}{c\varepsilon_0} \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$$

$$F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$$

Before we do that let us discuss the distinction between gauge symmetry (invariance) and usual symmetry:

Usual symmetry (Lorentz, translation, reflection...)

$\downarrow$

Transforms a solution of equations into a different solution [can be distinguished]

Gauge symmetry ($A_\mu \to A_\mu + \partial_\mu \alpha$)

$\downarrow$

Transforms a solution into physically identical, indistinguishable solution.

Back to field strength:

$$F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ is a tensor [4×4 table of numbers]

it is antisymmetric:

$$F_{\mu\nu} = - F_{\nu\mu}$$

We can raise and lower indices with

$\eta^{\mu\nu}$: $$F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\gamma} F_{\rho\gamma}$$

$F_{\mu\nu}$ transforms under Lorentz as a Lorentz tensor:

$$F_{\mu\nu} \longrightarrow \Lambda_\mu{}^\rho \Lambda_\nu{}^\gamma F_{\rho\gamma}(x^\mu(x'))$$

$$x^\mu \to x^{\mu'} = \Lambda^\mu{}_\nu x^\nu$$

$F_{\mu\nu}$ is gauge invariant:

$$A_\mu \to A_\mu + \partial_\mu \alpha$$

$$F_{\mu\nu} \to \partial_\mu A_\nu + \partial_\mu \partial_\nu \alpha -$$
$$- \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha = F_{\mu\nu}$$

- Now let us find the relation between $F_{\mu\nu}$ and $\vec{E}$ and $\vec{B}$:

remember that

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{E} = -\vec{\nabla} \cdot \varphi - \partial_t \vec{A}$$

$$A^\mu = (\varphi, c\vec{A}) \longrightarrow A_\mu = (-\varphi, c\vec{A})$$

$$F_{0i} = (0, \dot{A}_1 + \partial_1 \varphi, \dot{A}_2 + \partial_2 \varphi,$$
$$\dot{A}_3 + \partial_3 \varphi)$$

F_{ij} – ? Let's check

$$F_{21} = c\partial_2 A_1 - c\,\partial_1 A_2$$

$$B_3 = \varepsilon_{321}\partial_2 A_1 + \varepsilon_{312}\partial_1 A_2 =$$
$$= -\partial_2 A_1 + \partial_1 A_2 \Rightarrow$$

$F_{21} = -c\,B_3$. Check other components to see:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & cB_z & -cB_y \\ \text{Anti-sym.} & & 0 & cB_x \\ & & & 0 \end{pmatrix}$$

$$F_{23} = c\partial_2 A_3 - c\partial_3 A_2$$
$$B_1 = \varepsilon^{123}\partial_2 A_3 + \partial_3 A_2 \varepsilon^{132}$$

Covariant form of Maxwell equations

Let's remember Maxwell equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

First two equations take a very simple form:

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c\,\varepsilon_0} J^\nu \qquad (*)$$

It's simplest to check in Lorentz gauge:

$$-\partial_\mu \partial^\mu A^\nu + \underbrace{\cancel{\partial_\mu \partial^\nu A^\mu}}_{0} = \frac{1}{c\varepsilon_0} J^\nu$$

$$\Box A^\nu = \frac{1}{c\varepsilon_0} J^\nu$$

But (*) is gauge invariant, so it is true in every gauge!

Two other equations can be written as

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$$

where $\varepsilon^{\mu\nu\rho\sigma}$ is a 4-dimensional analog of ε^{ijk} (fully anti-symmetric tensor), $\varepsilon^{0123} = 1$

Again, if we substitute

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ then obviously

$$\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu\partial_\rho A_\sigma - \partial_\nu\partial_\sigma A_\rho) = 0$$

But it is a non-trivial constraint on $F_{\mu\nu}$ if we don't use A^μ.

- Next step:

We want to generalize Newton's laws to our Lorentz invariant framework.

It is natural to start with Lorentz force since we already know how the field that creates it transforms under Lorentz.

Remember that $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$.

Let us derive how $\frac{\partial}{\partial x_{\mu}} \equiv \partial_{\mu}$ transforms:

$$\frac{\partial}{\partial x^{\nu}} g(x) = \frac{\partial}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\nu}} = \frac{\partial}{\partial x'^{\mu}} \cdot \Lambda^{\mu}{}_{\nu} \Rightarrow$$

$$\Rightarrow \Lambda_{\rho}{}^{\nu} \partial_{\nu} = \Lambda_{\rho}{}^{\nu} \Lambda^{\mu}{}_{\nu} \partial'_{\mu}$$

Now let's prove that

$$\Lambda_{\rho}{}^{\nu} \Lambda^{\mu}{}_{\nu} \overset{(*)}{=} \eta_{\rho}{}^{\mu} \quad \text{(identity matrix)}$$

from this it will follow that

$$\partial'_{\mu} = \Lambda_{\rho}{}^{\nu} \partial_{\nu}.$$

From invariance of $\eta^{\mu\nu}$ we get:

$$\Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \eta^{\alpha\beta} = \eta^{\mu\nu} \Rightarrow \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu\alpha} = \eta^{\mu\nu} \Rightarrow$$

$$\Lambda_{\mu}{}^{\alpha} \Lambda^{\nu}{}_{\alpha} = \eta_{\mu}{}^{\nu} \Rightarrow (*)$$

- Infinitesemal interval

- Proper time:

$$d\tau = \frac{1}{c}\sqrt{-dx^\mu dx_\mu} = \sqrt{dt^2 - \frac{dx^2}{c^2}} = \frac{dt}{\gamma}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta\tau = \int_{t_1}^{t_2} dt \sqrt{1 - \frac{v^2}{c}}$$

4-velocity: $$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$$

$$u^\mu u_\mu = \frac{dx^\mu dx_\mu}{d\tau^2} = -c^2$$ [time-like vectors have negative]

Next we introduce 4-acceleration. Logic is the same: object that transforms as a vector and has the right $\beta \to 0$ limit.

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} \xrightarrow[\beta \ll 1]{} \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix}$$

Let's remember non-relativistic Lorentz force:

$$m\vec{a} = \vec{F} \qquad \frac{d\vec{p}}{dt} = \vec{F}$$

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

We would like an expression that reduces to it in $\gamma \to 1$ ~~limit~~.

In fact, to linear order in fields such expression is unique:

$$m^2 a^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$$

- It is a valid physical reasoning to derive the laws by symmetries!
- Now, spatial component:

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$\vec{p} = m\vec{v}\gamma$$

time component:

$$\frac{d\mathcal{E}}{dt} = q\vec{E} \cdot \vec{v} \quad [\text{work}]$$

$$\mathcal{E} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = \underline{\underline{mc^2}} + \frac{1}{2}mv^2$$

$m v^\mu$

$$p^\mu = \begin{pmatrix} \mathcal{E}/c \\ p \end{pmatrix}$$

four-vector of momentum

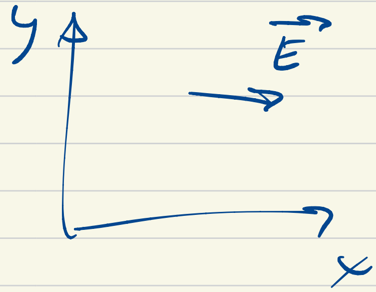
$$p^\mu p_\mu = m^2c^2$$

Applications:

- Constant electric field: $\vec{E} = \begin{pmatrix} E \\ 0 \\ 0 \end{pmatrix}$

$$p_x = qEt + p_x^0$$

$$p_y = p_y^0 = \text{const}$$



$$\mathcal{E} = \sqrt{m^2c^4 + p_y^{0\,2}c^2 + p_x^2c^2}$$

$$\vec{v} = c^2\frac{\vec{p}}{\mathcal{E}} \Rightarrow \frac{\partial x}{\partial t} = c^2\frac{qEt + p_x^0}{\sqrt{m^2c^4 + p_y^{0\,2}c^2 + (qEt + p_{x0})^2c^2}}$$

choose t, so that $p_x^0 = 0$.

then

$$\frac{\partial x}{\partial t} = \frac{c^2qEt}{\sqrt{\varkappa^2 + (qEt)^2c^2}} \leftarrow \varkappa^2 \equiv m^2c^4 + p_y^{0\,2}c^2$$

$$z = (cqEt)^2$$

$$x = \int^{x} dx = c\int^{t} dt \, \frac{cqEt\, dt}{\sqrt{\varkappa^2 + (cqEt)^2}} =$$

$$= \frac{c}{2cqE}\int^{z(t)} \frac{dz}{\sqrt{\varkappa^2 + z}} = \frac{1}{2qE}\, 2\sqrt{\varkappa^2 + z(t)} + X$$

int. const.

$$= \frac{1}{qE}\sqrt{\varkappa^2 + (cqEt)^2} + X_0$$ we found $x(t)$

note: $t \to \infty \quad x \simeq c \cdot t$

Inverting it we get $cqEt = \sqrt{(qE(x - X_0))^2 - \varkappa^2}$

Now, let us find the trajectory

$$\frac{dx}{dy} = \frac{dx}{dt} \Big/ \frac{dy}{dt}$$

$$\frac{\partial x}{\partial t} = \frac{c^2 q E t}{\sqrt{\varkappa^2 + (qEt)^2 c^2}}$$

$$\frac{\partial y}{\partial t} = c^2 \frac{p_y^0}{\sqrt{\varkappa^2 + (qEt)^2 c^2}}$$

$$\frac{dx}{dy} = \frac{cqEt}{cp_y^0} = \frac{1}{cp_y^0}\sqrt{(qE(x - X_0))^2 - \varkappa^2}$$

$$dy = cp_y^0 \frac{dx}{\sqrt{(qE(x - X_0))^2 - \varkappa^2}}$$

$$\omega \equiv qE(x - X_0)$$

$$\frac{d\omega}{\sqrt{\omega^2 - \varkappa^2}} = \frac{dy\, qE}{cp_y^0}$$

$$\omega = \varkappa \cosh \alpha$$

$$\frac{\varkappa \sinh\alpha \, d\alpha}{\sqrt{\varkappa^2(\cosh^2\alpha - 1)}} \approx dy$$

$$d\alpha \approx dy \Rightarrow \alpha \approx y + \text{const}$$

$$qE(x - X_0) = \varkappa(y + \text{const})$$

$$\boxed{x(y) = \frac{\sqrt{m^2c^4 + p_y^{0\,2}c^2}}{qE}\left(\cosh\left(\frac{qEy}{cp_y^0}\right) - 1\right)}$$

End of lecture 13

Magnetic field.

$$\frac{d\vec{p}}{dt} = q(\vec{v} \times \vec{B})$$

$$\frac{d}{dt}\mathcal{E} = 0 \Rightarrow \text{const} \Rightarrow |\vec{v}| = \text{const}$$

$$\frac{d\vec{v}}{dt} = \vec{v} \times \vec{\omega}_B, \qquad \omega_B = \frac{q\vec{B}}{\gamma m} = \frac{q\vec{B}c^2}{\mathcal{E}}$$

this is called "cyclotron frequency"

$B \parallel B_z$

$$\frac{dv_x}{dt} = v_y \omega_B \qquad \frac{dv_y}{dt} = -v_x \omega_B$$

Relativistic generalization of Larmor formula.

$$P = \frac{q^2}{4\pi\varepsilon_0} \dot{v}^2 = \frac{d\varepsilon}{dt}$$

$$dp^\mu = -\frac{q^2}{6\pi\varepsilon_0 c^3}\left(\frac{du^\nu}{d\tau}\frac{du_\nu}{d\tau}\right) dx^\mu$$

$$P = \frac{dP^0}{dx^0} = -\frac{q^2}{6\pi\varepsilon_0 c}\left(\frac{du^\nu du_\nu}{d\tau\, d\tau}\right) =$$

$$= \frac{q^2}{6\pi\varepsilon_0 c}\gamma^6\left[|\dot{\beta}|^2 - |\beta \times \dot{\beta}|^2\right]$$